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EFFECT OF RANDOM MATERIAL PARAMETERS ON NONLINEAR STEADY CREEP SOLUTIONS†

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Abstract—An analysis is presented concerning the effect of random material parameters on nonlinear steady creep in a 3-bar truss. Parameter randomness is in general large and is introduced through randomness in the temperature and in the density of imperfections. Analytical and numerical results are presented on the statistical properties of the material parameters and of the stress and velocity. It is shown that randomness in the material parameters will on the one hand introduce only a very slight randomness in stress and on the other hand a very significant randomness in velocity.

1. INTRODUCTION

THE scatter of data obtained from even carefully conducted creep tests on metals at elevated temperatures is far greater than that for ordinary room temperature testing. Such scatter is of course even more pronounced under actual service conditions. It is due to random fluctuations in the load and temperature and to the presence of random material imperfections (Hoff [1]). Randomness in the load results in a "random input problem", and such problems are not the concern of this work. On the other hand, since the creep parameters are highly dependent on temperature, randomness in the temperature does result in a "random parameter problem". Some work in this area has been published (Soong and Cozzarelli [2], Parkus [3]), and it has been found to be a significant effect; we shall consider this effect in the present work. The effect of random material imperfections on nonlinear steady creep solutions is a random parameter problem which has received virtually no attention to date, and thus we turn our attention to this problem as well.

If one can obtain accurate control over the environment during creep testing and during actual service conditions, then the effect of randomness in load can be minimized. On the other hand, it is more difficult to control the temperature so that, for example, it is actually uniform. Furthermore, material imperfections are microscopic in nature and it is virtually impossible to obtain effective control of such imperfections. Various kinds of imperfections, such as dislocations, point defects, impurities, grain boundaries, etc. are always present in a metal, and in numbers which are for the most part unknown. The creep parameters are highly temperature and structure sensitive, and accordingly they vary greatly from sample to sample and from point to point within a sample. Thus, it would

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appear that randomness in the material parameters is probably the most significant source of the observed randomness in a creeping metal structure.

It is well known that for creeping metals the strain depends not only on time but also exhibits a nonlinear dependence on stress. If we hope to obtain useful statistical results from such a complicated expression, we must confine ourselves, at least in the beginning, to the simplest possible situation. Thus, we will assume that creep has entered the secondary creep stage and that the steady creep strain dominates the elastic and transient creep strains. In such a situation we may employ the widely used creep power law, which in the one-dimensional case may be written as

$$\dot{\varepsilon} = S\sigma^n. \tag{1}$$

Here, σ is the stress, $\dot{\varepsilon}$ is the strain rate and S and n are material parameters.

Now as we have already remarked, even if the load may be considered deterministic, randomness in the temperature and the imperfection density and consequently in the material parameters is still generally present. Thus, we consider the parameters S and n to be random parameters $\hat{S}$ and $\hat{n}$, where a carat over a symbol is used here and in the sequel to represent a random quantity. Clearly, in a structure governed by equation (1) the strain rate is a function of these random parameters and thus $\dot{\varepsilon}$ is also a random quantity $\hat{\dot{\varepsilon}}$. Furthermore, if the structure is statically indeterminate the stress will be a function of $\hat{n}$, and thus σ will also be a random quantity $\hat{\sigma}$. Accordingly, equation (1) must in general be written as

$$\hat{\dot{\varepsilon}} = \hat{S}\hat{\sigma}^{\hat{n}}. \tag{2}$$

Equation (2) is not in a convenient form, since the units of $\hat{S}$ contain the random power $\hat{n}$ and are thus random. Following Odqvist [4], we shall rewrite this equation as

$$\hat{\dot{\varepsilon}} = \hat{\dot{\varepsilon}}_c\left(\frac{\hat{\sigma}}{\sigma_c}\right)^{\hat{n}} \tag{3}$$

where σ_c is an arbitrary deterministic constant equal to a reference creep stress, and $\hat{\dot{\varepsilon}}_c$ is a random material parameter equal to the random creep rate obtained when $\hat{\sigma} = \sigma_c$. The units of $\hat{\dot{\varepsilon}}_c$ are deterministic, e.g. $(\text{hr})^{-1}$.

In general we might expect that the parameters $\hat{\dot{\varepsilon}}_c$ and $\hat{n}$ would depend on both the temperature $\hat{T}$ and the imperfection density $\hat{N}$, and accordingly $\hat{\dot{\varepsilon}}_c$ and $\hat{n}$ would be correlated random quantities. However, experimental evidence (e.g. see Dorn [5]) indicates that $\hat{T}$ has a negligible effect on $\hat{n}$ and conversely $\hat{N}$ has a negligible effect on $\hat{\dot{\varepsilon}}_c$. Thus we may as a first approximation write

$$\hat{\dot{\varepsilon}}_c = F(\hat{T}) \qquad \hat{n} = G(\hat{N}) \tag{4}$$

where, in accordance with experiment, $F(\hat{T})$ is a monotonically increasing function and $G(\hat{N})$ is monotonically decreasing. It is clear on physical grounds that $\hat{T}$ and $\hat{N}$ may be considered independent, and it follows from equations (4) that $\hat{\dot{\varepsilon}}_c$ and $\hat{n}$ will also be independent.

The present paper considers the specific problem of creep in a 3-bar truss subjected to a prescribed deterministic load, where the creep parameters $\hat{\dot{\varepsilon}}_c$ and $\hat{n}$ are random processes in space resulting from random processes $\hat{T}$ and $\hat{N}$. The results of the analysis of this particular statically indeterminate creep problem will serve to provide insight into the effect of random parameters on creep in indeterminate structures in general. The derivation of the governing expressions for the stresses and velocities for this truss, with arbitrary statistics for $\hat{\dot{\varepsilon}}_c$ and $\hat{n}$, is presented in Section 2. This is followed in Section 3 by a discussion of a probabilistic model based on equation (4), with $\hat{T}$ and $\hat{N}$ assumed to be independent homogeneous random processes. It is shown that for large randomness the first order density functions of both $\hat{\dot{\varepsilon}}_c$ and $\hat{n}$ are lognormal. Then, the density functions and statistical moments of the stresses and velocities are given in Section 4. The final section contains a summary of the results obtained.

2. GOVERNING EQUATIONS

Consider a 3-bar truss subjected to a prescribed deterministic vertical load Q at point o as shown in Fig. 1. The cross-sectional area $\bar{A}$ is taken constant and equal for all bars, but the temperature and imperfection density and accordingly the material parameters are random functions of the distance along the bars. We see from Fig. 1 that the coordinate y defines points on each of the three bars, and that in particular $y = 0$ defines junction point o and $y = -l$ defines the three supports. Using the subscript $i = 1, 2, 3$ to identify a particular bar, we obtain from stress–strain relation (3)

$$\hat{\dot{\varepsilon}}_i(y) = \hat{\dot{\varepsilon}}_{ci}(y)\left[\frac{\hat{\sigma}_i(y)}{\sigma_c}\right]^{\hat{n}_i(y)} \qquad i = 1, 2, 3 \tag{5}$$

where the material parameters $\hat{\dot{\varepsilon}}_{ci}(y)$ and $\hat{n}_i(y)$, the axial stress $\hat{\sigma}_i(y)$ and the axial strain $\hat{\dot{\varepsilon}}_i(y)$ in the ith bar are in general random processes in y.

Now, equilibrium in the ith bar yields the condition

$$\hat{\sigma}_i(y) = \hat{\sigma}_i(0) = \hat{\sigma}_i. \tag{6}$$

Thus we see that the stresses $\hat{\sigma}_i$ are independent of y and hence are simply random variables. Then, equilibrium at point o gives

$$\hat{\sigma}_1 = \hat{\sigma}_3 \tag{7a}$$

$$\hat{\sigma}_2 + 2\hat{\sigma}_1 \sin\theta = Q/\bar{A} = \sigma_c \tag{7b}$$

where θ is the angle to bars 1 and 3, and where the reference stress σ_c in equation (5) has been chosen here as $Q/\bar{A}$.

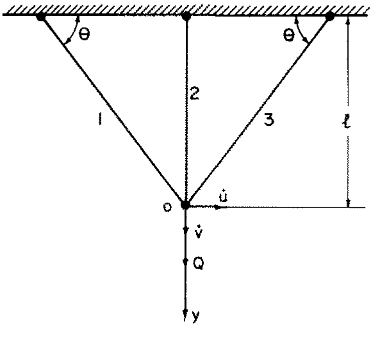

FIG. 1. 3-bar truss.

Next, geometric compatibility at point o requires

$$\hat{u}\cos\theta+\hat{v}\sin\theta = \hat{U}_1(0) \tag{8a}$$

$$\hat{v} = \hat{U}_2(0) \tag{8b}$$

$$-\hat{u}\cos\theta+\hat{v}\sin\theta = \hat{U}_3(0) \tag{8c}$$

where $\hat{u}$ and $\hat{v}$ are the horizontal and vertical velocity components respectively of the junction point o (see Fig. 1), and where $\hat{U}_1(0)$, $\hat{U}_2(0)$ and $\hat{U}_3(0)$ are the axial velocities in the three bars at point o. Equations (8) are three equations in five velocity variables, and thus only two of these velocity variables are independent. For example, if $\hat{U}_1(0)$ and $\hat{U}_2(0)$ have been determined, then in accordance with equations (8) $\hat{U}_3(0)$ must satisfy the compatibility equation

$$\hat{U}_1(0)+\hat{U}_3(0)-2\hat{U}_2(0)\sin\theta = 0 \tag{9}$$

and $\hat{u}$ and $\hat{v}$ follow directly from equations (8a) and (8b).

Finally, we have the strain displacement relations

$$\begin{aligned}\hat{\varepsilon}_1(y) &= \frac{\mathrm{d}\hat{U}_1(y)}{\mathrm{d}(y/\sin\theta)} = \sin\theta\frac{\mathrm{d}\hat{U}_1(y)}{\mathrm{d}y}\\ \hat{\varepsilon}_2(y) &= \frac{\mathrm{d}\hat{U}_2(y)}{\mathrm{d}y}\\ \hat{\varepsilon}_3(y) &= \sin\theta\frac{\mathrm{d}\hat{U}_3(y)}{\mathrm{d}y}.\end{aligned} \tag{10}$$

Using equations (5) and (6) to eliminate the strains from equations (10), and integrating from $y = -l$ to $y = 0$ with boundary conditions $\hat{U}_1(-l) = \hat{U}_2(-l) = \hat{U}_3(-l) = 0$, yields axial velocities at o

$$\hat{U}_1(0) = \csc\theta\int_0^l \hat{\varepsilon}_{c1}(y)\left(\frac{\hat{\sigma}_1}{\sigma_c}\right)^{\hat{n}_1(y)}\mathrm{d}y \tag{11a}$$

$$\hat{U}_2(0) = \int_0^l \hat{\varepsilon}_{c2}(y)\left(\frac{\hat{\sigma}_2}{\sigma_c}\right)^{\hat{n}_2(y)}\mathrm{d}y \tag{11b}$$

$$\hat{U}_3(0) = \csc\theta\int_0^l \hat{\varepsilon}_{c3}(y)\left(\frac{\hat{\sigma}_3}{\sigma_c}\right)^{\hat{n}_3(y)}\mathrm{d}y. \tag{11c}$$

By replacing the lower limit 0 in equations (11) with variable limit y, we may obtain corresponding expressions for the random processes $\hat{U}_i(y)$.

A single stress compatibility equation in the random variable $\hat{\sigma}_2$ may be obtained by substituting equations (11) into equation (9) with $\hat{\sigma}_1$ and $\hat{\sigma}_3$ eliminated by means of equations (7), giving

$$\int_0^l\left[\hat{\varepsilon}_{c1}(y)\left(\frac{\sigma_c-\hat{\sigma}_2}{2\sigma_c\sin\theta}\right)^{\hat{n}_1(y)}+\hat{\varepsilon}_{c3}(y)\left(\frac{\sigma_c-\hat{\sigma}_2}{2\sigma_c\sin\theta}\right)^{\hat{n}_3(y)}-2\sin^2\theta\hat{\varepsilon}_{c2}(y)\left(\frac{\hat{\sigma}_2}{\sigma_c}\right)^{\hat{n}_2(y)}\right]\mathrm{d}y = 0. \tag{12}$$

Once this integral equation has been solved for $\hat{\sigma}_2$, we may obtain $\hat{\sigma}_1$ and $\hat{\sigma}_3$ from equations (7), $\hat{U}_1(0)$ and $\hat{U}_2(0)$ from equations (11a) and (11b), $\hat{U}_3(0)$ from equation (9), and finally $\hat{u}$ and $\hat{v}$ from equations (8a) and (8b).

As we have stated, the randomness in the material parameters will in general be large. Accordingly, the use of a perturbation technique which requires that the parameters be slightly random is precluded in the present case. We are thus faced with a very difficult nonlinear problem, and it is clear that some simplification is essential. Hence, we shall in the present analysis take $\theta = 45°$ and make the rather strong assumption that the temperature and imperfection density and accordingly the material parameters are equal random processes in the three bars, i.e.

$$\hat{n}_1(y) = \hat{n}_2(y) = \hat{n}_3(y) = \hat{n}(y) \tag{13a}$$

$$\hat{\varepsilon}_{c1}(y) = \hat{\varepsilon}_{c2}(y) = \hat{\varepsilon}_{c3}(y) = \hat{\varepsilon}_c(y). \tag{13b}$$

Then, we obtain from the above formulation

$$\int_0^l \hat{\varepsilon}_c(y)\left[\left(\frac{\sigma_c-\hat{\sigma}_2}{\sqrt{2}\sigma_c}\right)^{\hat{n}(y)} - \frac{1}{2}\left(\frac{\hat{\sigma}_2}{\sigma_c}\right)^{\hat{n}(y)}\right] \mathrm{d}y = 0 \tag{14a}$$

$$\hat{u} = 0 \qquad \hat{v} = \int_0^l \hat{\varepsilon}_c(y)\left(\frac{\hat{\sigma}_2}{\sigma_c}\right)^{\hat{n}(y)} \mathrm{d}y. \tag{14b}$$

In the special case of material parameters independent of y (i.e. random variables), equations (14) simplify further to

$$\hat{\sigma}_2 = \frac{\sigma_c}{1+2^{(\hat{n}-2)/2\hat{n}}} \tag{15a}$$

$$\hat{v} = l\hat{\varepsilon}_c\left(\frac{\hat{\sigma}_2}{\sigma_c}\right)^{\hat{n}} = \frac{l\hat{\varepsilon}_c}{(1+2^{(\hat{n}-2)/2\hat{n}})^{\hat{n}}}. \tag{15b}$$

In the following section we make some plausible assumptions about the statistics for the temperature $\hat{T}(y)$ and imperfection density $\hat{N}(y)$, and then develop the statistics of parameters $\hat{\varepsilon}_c(y)$ and $\hat{n}(y)$ from functions F and G in equations (4).

3. PROBABILISTIC MODEL

We assume that the imperfection density $\hat{N}(y)$ and the temperature $\hat{T}(y)$ are homogeneous normal random processes. The first order density functions are then given by

$$f(N) = \frac{1}{\sqrt{(2\pi)}\sigma_N} \mathrm{e}^{-(N-N_0)^2/2\sigma_N^2} \tag{16a}$$

$$f(T) = \frac{1}{\sqrt{(2\pi)}\sigma_T} \mathrm{e}^{-(T-T_0)^2/2\sigma_T^2} \tag{16b}$$

where σ_N^2, σ_T^2 are the variances and N_0, T_0 are the means. Since the randomness in $\hat{T}(y)$ is essentially unrelated to the randomness in $\hat{N}(y)$, it is reasonable that $\hat{N}(y)$ and $\hat{T}(y)$ be considered independent.

We may demonstrate the plausibility of equation (16a) by considering the following reasonable model for the introduction of imperfections into a metallic bar during manufacture. We are given m imperfections and M metallic bars, where $m \gg M$. Our experiment consists of dividing these imperfections among the bars by "tossing" them into these bars one at a time and at random, where for the moment we think of each bar as a "box" and ignore the space coordinate y. We may assume that the trials are independent because there are a very large number of lattice sites in a particular bar into which an imperfection may fall. This is a classic problem in probability [6], and the probability density function for the number of imperfections in a particular bar is very closely approximated by equation (16a), with $\sigma_N^2 = (m/M)(1-1/M) \gg 1$ and $N_0 = m/M$. Having thus demonstrated that it is reasonable to take $\hat{N}$ as a normal random variable, it is logical to extend the argument further by taking $\hat{N}(y)$ as a normal random process.

Turning now to the temperature, we may demonstrate the reasonableness of equation (16b) by constructing a similar experiment. Here we let m equal the number of impulses of heat that the surrounding medium transmits to the M bars by random impact. By making assumptions analogous to those made above, we may arrive at equation (16b).

Now we may develop the statistics of parameters $\hat{\dot{\varepsilon}}_c(y)$ and $\hat{n}(y)$ in accordance with equations (4).

3(a). *Statistics of* $\hat{\dot{\varepsilon}}_c(y)$

The most widely quoted [5] form of the function F in equations (4) is

$$\hat{\dot{\varepsilon}}_c(y) = F[\hat{T}(y)] = A\,\mathrm{e}^{-B/\hat{T}(y)} \tag{17}$$

where A and B are taken as temperature and imperfection insensitive material constants. However, as discussed in [4, 7], equation (17) may be approximated with good accuracy by the more convenient expression

$$\hat{\dot{\varepsilon}}_c(y) = \dot{\varepsilon}_{c0}\,\mathrm{e}^{\hat{\tau}(y)} \tag{18a}$$

$$\hat{\tau}(y) = \frac{B[\hat{T}(y) - T_0]}{T_0^2} \tag{18b}$$

where T_0 is a constant reference temperature taken here as the mean value, and $\dot{\varepsilon}_{c0}$ is the value of $\hat{\dot{\varepsilon}}_c$ at $\hat{T} = T_0$ which shall be called here the nominal value.

Since $\hat{T}(y)$ is homogeneous normal with mean T_0 [equation (16b)], it follows from equation (18b) that the nondimensional temperature $\hat{\tau}(y)$ is homogeneous normal with mean zero, i.e. its first order density function is

$$f(\tau) = \frac{1}{\sqrt{(2\pi)}\sigma_\tau}\,\mathrm{e}^{-\tau^2/2\sigma_\tau^2} \tag{19}$$

where σ_τ^2 is the variance of $\hat{\tau}(y)$. Rewriting equation (18a) as

$$\hat{\mathscr{E}}(y) = \frac{\hat{\dot{\varepsilon}}_c(y)}{\dot{\varepsilon}_{c0}} = \mathrm{e}^{\hat{\tau}(y)} \tag{20}$$

we may then easily show from equations (19) and (20) that the nondimensional parameter $\hat{\mathscr{E}}(y)$ is a homogeneous random process with first order density function

$$f(\mathscr{E}) = \frac{1}{\sqrt{(2\pi)}\sigma_\tau \mathscr{E}}\,\mathrm{e}^{-(\ln \mathscr{E})^2/2\sigma_\tau^2}\,U(\mathscr{E}) \tag{21}$$

where $U(\mathscr{E})$ is the unit step function.

Equation (21) is the lognormal distribution [8], and its properties are well known. In particular, the mean $E\{\hat{\mathscr{E}}\}$, variance $\sigma_{\mathscr{E}}^2$ and most probable value $\max\{\hat{\mathscr{E}}\}$ are given by

$$E\{\hat{\mathscr{E}}\} = e^{\sigma_\tau^2/2} > 1 \tag{22a}$$

$$\sigma_{\mathscr{E}}^2 = e^{\sigma_\tau^2}(e^{\sigma_\tau^2}-1) > \sigma_\tau^2 \tag{22b}$$

$$\max\{\hat{\mathscr{E}}\} = e^{-\sigma_\tau^2} < 1. \tag{22c}$$

By considering second order statistics we find the autocovariance

$$C_{\mathscr{E}}(\eta) = e^{\sigma_\tau^2}(e^{C_\tau(\eta)}-1) \tag{23}$$

where $C_\tau(\eta)$ is the autocovariance of $\hat{\tau}(y)$ and $\eta = y_1 - y_2$. In subsequent work, we will choose for $C_\tau(\eta)$ the physically plausible exponential expression

$$C_\tau(\eta) = \sigma_\tau^2\, e^{-|\eta|/d} \tag{24}$$

where d is the correlation distance. For this case, we will prove in Section 4(a).2 that $\hat{\mathscr{E}}(y)$ and correspondingly $\hat{\varepsilon}_c(y)$ are ergodic in the mean.

Finally, one can show that if $\hat{\mathscr{E}}$ is distributed near to its nominal value 1 (value at $\hat{T} = T_0$) it is then approximately normal with mean 1 and variance σ_τ^2, i.e.

$$f(\mathscr{E}) \approx \frac{1}{\sqrt{(2\pi)}\sigma_\tau} e^{-(\mathscr{E}-1)^2/2\sigma_\tau^2}, \qquad \sigma_\tau \text{ small.} \tag{25}$$

The lognormal distribution for $\hat{\mathscr{E}}$ [equation (21)] and its normal approximation [equation (25)] are illustrated in Fig. 2 for two values of σ_τ, where the left-hand and lower scales apply in this case.

3(b). *Statistics of* $\hat{n}(y)$

The exact form of the function $G(\hat{N})$ in equations (4) is not as well established as the form of function $F(\hat{T})$. However, from the discussion and data given by Garafalo [9], we

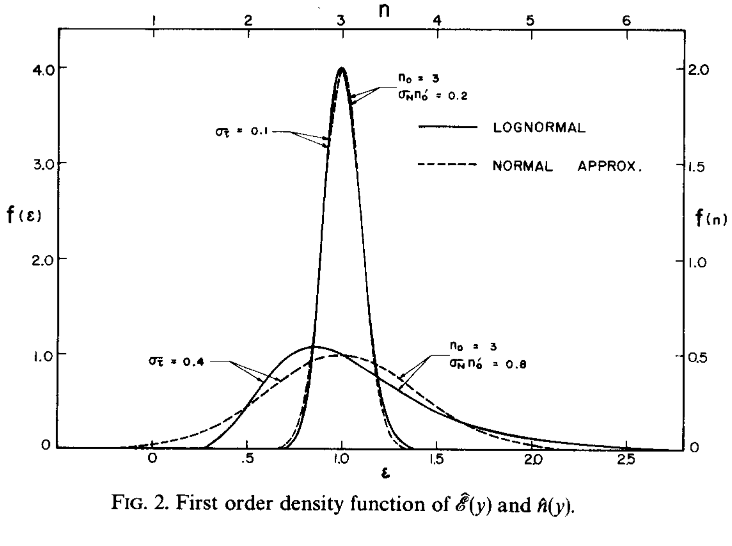

FIG. 2. First order density function of $\hat{\mathscr{E}}(y)$ and $\hat{n}(y)$.

conclude that $\hat{n}(y)$ decreases in an exponential-like manner as the imperfection density $\hat{N}(y)$ increases. Furthermore, $\hat{n}(y) \to 1$ as $\hat{N}(y) \to \infty$, i.e. a creeping metal tends toward a Newtonian fluid as the structure tends toward complete disorder. Based on these observations we propose the relation

$$\hat{n}(y) = G[\hat{N}(y)] = 1 + \alpha\, e^{-\beta \hat{N}(y)} \tag{26}$$

where α and β are imperfection and temperature insensitive material constants.

The constants α and β in equation (26) are conveniently expressed in terms of data at the mean value N_0. Equation (26) may then be rewritten as

$$\hat{n}(y) = 1 + (n_0 - 1)\exp\left(-\frac{n_0'[\hat{N}(y) - N_0]}{(n_0 - 1)}\right) \tag{27}$$

where n_0 and the negative of the slope n_0' are nominal values defined as

$$n_0 = \hat{n}|_{\hat{N}=N_0} \qquad n_0' = -\mathrm{d}\hat{n}/\mathrm{d}\hat{N}|_{\hat{N}=N_0}. \tag{28}$$

Now, $\hat{N}(y)$ has been assumed to be homogeneous normal with mean N_0 and its first order density function is given by equation (16a). It then follows from equation (27) that $\hat{n}(y)$ is a homogeneous random process with first order density function

$$f(n) = \frac{(n_0 - 1)}{\sqrt{(2\pi)}\sigma_N n_0'(n-1)} \exp\left\{-\frac{(n_0-1)^2}{2(\sigma_N n_0')^2}\left[\ln\left(\frac{n-1}{n_0-1}\right)\right]^2\right\} U(n-1). \tag{29}$$

This is a lognormal distribution [8] with two parameters (n_0 and $\sigma_N n_0'$), and $\hat{n}$ has the following statistical properties:

$$E\{\hat{n}\} = 1 + (n_0 - 1)\exp\left[\frac{(\sigma_N n_0')^2}{2(n_0-1)^2}\right] > n_0 \tag{30a}$$

$$\sigma_n^2 = (n_0 - 1)^2 \exp\left[\frac{(\sigma_N n_0')^2}{(n_0-1)^2}\right]\left\{\exp\left[\frac{(\sigma_N n_0')^2}{(n_0-1)^2}\right] - 1\right\} > (\sigma_N n_0')^2 \tag{30b}$$

$$\max\{\hat{n}\} = 1 + (n_0 - 1)\exp\left[-\frac{(\sigma_N n_0')^2}{(n_0-1)^2}\right] < n_0. \tag{30c}$$

Note that $f(n)$ starts at zero at $n = 1$, builds up to a peak to the left of the nominal value n_0 [equation (30c)], and then tapers in a tail to infinity.

We can easily determine the autocovariance

$$C_n(\eta) = (n_0 - 1)^2 \exp\left[\frac{(\sigma_N n_0')^2}{(n_0-1)^2}\right]\left\{\exp\left[\frac{n_0'^2}{(n_0-1)^2} C_N(\eta)\right] - 1\right\} \tag{31}$$

where $C_N(\eta)$ is the autocovariance of $\hat{N}(y)$. If $C_N(\eta)$ is chosen as the exponential expression

$$C_N(\eta) = \sigma_N^2\, e^{-|\eta|/d} \tag{32}$$

then analogous to the behavior of $\hat{\mathscr{E}}(y)$ one can show that $\hat{n}(y)$ is ergodic in the mean [6].

Also, if $\sigma_N n_0'/(n_0 - 1)$ is small enough (e.g. $\leq 0{\cdot}2$) we may approximate equation (29) by the normal distribution

$$f(n) \approx \frac{1}{\sqrt{(2\pi)}\sigma_n} \exp\left(-\frac{(n-n_0)^2}{2\sigma_n^2}\right), \qquad \frac{\sigma_N n_0'}{n_0 - 1} \text{ small} \tag{33}$$

where $E\{\hat{n}\} = n_0$ and $\sigma_n^2 = (\sigma_N n_0')^2$. Figure 2 may be used again to illustrate the lognormal distribution for $\hat{n}$ [equation (29)] and its normal approximation [equation (33)], where now the right-hand and upper scales apply.

In the next section, we return to the 3-bar truss problem formulated in Section 2 and utilize the probabilistic model presented here in conjunction with equations (14) to develop the statistics of the stresses and velocities.

4. STATISTICAL ANALYSIS OF STRESS AND VELOCITY

In accordance with our development, the random temperature $\hat{T}(y)$ affects only the parameter $\hat{\dot{\varepsilon}}_c(y)$ whereas the random imperfection density $\hat{N}(y)$ affects only the parameter $\hat{n}(y)$. And, $\hat{T}(y)$ and $\hat{N}(y)$ are independent. Hence, the "random temperature problem" [equations (14) plus equations (18)] and the "random imperfection problem" [equations (14) plus equation (27)] are uncoupled problems, and we treat them separately.

4(a). *Random temperature problem*

1. *Stress solution.* If only the temperature $\hat{T}(y)$ is random we set $\hat{n}(y) = n_0$, and since $\int_0^l \hat{\dot{\varepsilon}}_c(y)\,\mathrm{d}y > 0$ equation (14a) yields the following simple solution for stress:

$$\hat{\sigma}_2 = \frac{\sigma_c}{1+2^{(n_0-2)/2n_0}} = \sigma_2. \tag{34}$$

Thus, stresses $\hat{\sigma}_1, \hat{\sigma}_2$ and $\hat{\sigma}_3$ [equations (7)] are all deterministic constants. Had we not assumed $\hat{\dot{\varepsilon}}_{c1}(y) = \hat{\dot{\varepsilon}}_{c2}(y) = \hat{\dot{\varepsilon}}_{c3}(y)$, equation (12) would have given the solution

$$\hat{\sigma}_2 = \frac{\sigma_c}{1+\sqrt{2}\left[\dfrac{\int_0^l \hat{\dot{\varepsilon}}_{c2}(y)\,\mathrm{d}y}{\int_0^l \hat{\dot{\varepsilon}}_{c1}(y)\,\mathrm{d}y+\int_0^l \hat{\dot{\varepsilon}}_{c3}(y)\,\mathrm{d}y}\right]^{1/n_0}}. \tag{35}$$

Now, in accordance with our previous discussion the $\hat{\dot{\varepsilon}}_{ci}(y)$ in the 3-bars are homogeneous random processes with equal ergodic mean, whereupon equation (35) reduces to equation (34) as $l \to \infty$ and again the stresses are deterministic. Thus, we may expect that, even under the most general of conditions, randomness in temperature will introduce only a very slight randomness in stress.

2. *Velocity solution.* Turning now to the velocity, we obtain from equations (14b) [with $\hat{n}(y) = n_0$ and $\hat{\sigma}_2 = \sigma_2$] and definition (20) the expressions

$$\left(\frac{\hat{\dot{v}}}{\dot{v}_0}\right) = \frac{1}{l}\int_0^l \hat{\mathscr{E}}(y)\,\mathrm{d}y \tag{36a}$$

$$\dot{v}_0 = \dot{\varepsilon}_{c0}\left(\frac{\sigma_2}{\sigma_c}\right)^{n_0} l \tag{36b}$$

where $\dot{v}_0$ is the nominal velocity (value at $\hat{T} = T_0$). We see that for the special case $\hat{\mathscr{E}}(y) = \hat{\mathscr{E}}$ (random variable) the density function $f(\dot{v}/\dot{v}_0)$ is identical with the first order density function $f(\mathscr{E})$ [equation (21)], i.e. it is lognormal. In the general case of $\hat{\mathscr{E}}(y)$ a homogeneous random process, the determination of the density function of the velocity ratio $\hat{v}/\dot{v}_0$ is very complicated and shall not be attempted here. However, we shall determine the mean and variance of $\hat{v}/\dot{v}_0$.

The mean follows directly from equations (36a) and (22a) and the homogeneity of $\hat{\mathscr{E}}(y)$ as

$$E\{\hat{v}/\dot{v}_0\} = E\{\hat{\mathscr{E}}(y)\} = e^{\sigma_{\hat{\tau}}^2/2} > 1 \tag{37}$$

whereupon we see that the mean velocity always exceeds the nominal velocity. It may be shown [6] that the variance is given by integral

$$\sigma^2_{\hat{v}/\dot{v}_0} = \frac{1}{l}\int_0^{2l}\left(1-\frac{\eta}{2l}\right)C_{\mathscr{E}}(\eta)\,\mathrm{d}\eta \tag{38}$$

where the autocovariance $C_{\mathscr{E}}(\eta)$ must satisfy equation (23). For the purpose of illustration, we choose equation (24) for the autocovariance of $\hat{\tau}(y)$. Substituting this equation into equation (23) we get

$$C_{\mathscr{E}}(\eta) = e^{\sigma_{\hat{\tau}}^2}[\exp(\sigma_\tau^2\, e^{-|\eta|/d}) - 1] \tag{39}$$

and then evaluating integral (38) we obtain the expression

$$\frac{\sigma^2_{\hat{v}/\dot{v}_0}}{\sigma^2_{\mathscr{E}}} = \frac{1}{e^{\sigma_{\hat{\tau}}^2}-1}\,\frac{d}{l}\sum_{k=1}^{\infty}\frac{\sigma_\tau^{2k}}{k\,.\,k!}\left[1-\frac{d}{2kl}(1-e^{-2kl/d})\right]. \tag{40}$$

The following limiting values follow from this result:

$$\lim_{d/l\to 0}\frac{\sigma^2_{\hat{v}/\dot{v}_0}}{\sigma^2_{\mathscr{E}}} = 0 \qquad \lim_{d/l\to\infty}\frac{\sigma^2_{\hat{v}/\dot{v}_0}}{\sigma^2_{\mathscr{E}}} = 1. \tag{41}$$

Physically, the limit $d/l \to \infty$ (perfect correlation) is equivalent to assuming that $\hat{\tau}(y)$ and $\hat{\mathscr{E}}(y)$ are simply random variables, while the limit $d/l \to 0$ (uncorrelated) corresponds with taking $\hat{\tau}(y)$ as white noise. Finally, we note that since $\hat{v}/\dot{v}_0$ is the spatial average of $\hat{\mathscr{E}}(y)$ [equation (36a)] the vanishing of $\sigma^2_{\hat{v}/\dot{v}_0}$ as $l \to \infty$ verifies our earlier statement that $\hat{\mathscr{E}}(y)$ is ergodic in the mean when equation (24) is valid.

We note from the above that both the variance and the mean of the velocity ratio $\hat{v}/\dot{v}_0$ are independent of the parameter n_0. In Fig. 3, the variance ratio given by equation (40)

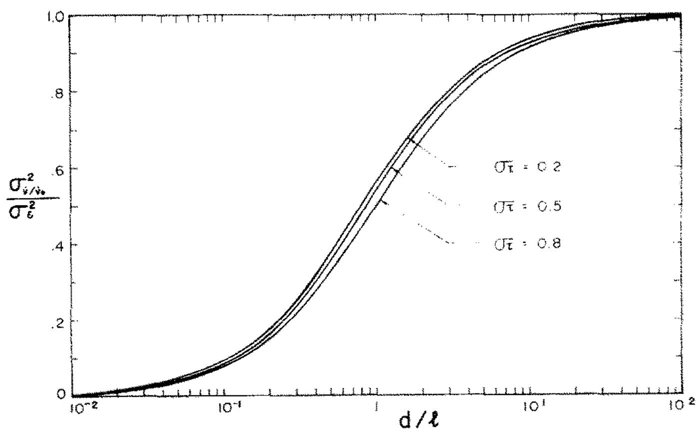

FIG. 3. Variance of velocity—random temperature problem.

has been plotted vs. the distance ratio d/l for several values of σ_τ. The velocity is clearly very sensitive to variations in temperature. For example, consider the typical values $T_0 = 1000°\text{R}$, $B = 30000°\text{R}$, $\sigma_T = 10°$ [i.e. $P\{970 \leq \hat{T} \leq 1030\} = 0{\cdot}9973$] and $d/l = 1$; equations (18b) and (22b) then yield $\sigma_\tau = 0{\cdot}3$ and $\sigma_{\mathscr{E}} = 0{\cdot}321$. And, equations (37) and (40) yield the values $E\{\hat{v}/\dot{v}_0\} = 1{\cdot}046$ and $\sigma_{\dot{v}/\dot{v}_0} = 0{\cdot}240$—clearly a significant effect. The results obtained in this section are in qualitative agreement with the results obtained for a circular plate in [2].

4(b). *Random imperfection problem*

1. *Stress solution.* In this problem we set $\hat{\dot{\varepsilon}}_c(y) = \dot{\varepsilon}_{c0}$, whereupon equation (14a) becomes

$$\int_0^l \left[\left(\frac{\sigma_c-\hat{\sigma}_2}{\sqrt{(2)}\sigma_c}\right)^{\hat{n}(y)} - \frac{1}{2}\left(\frac{\hat{\sigma}_2}{\sigma_c}\right)^{\hat{n}(y)}\right] \mathrm{d}y = 0. \tag{42}$$

In principle, this integral equation specifies the statistics of random variable $\hat{\sigma}_2$ in terms of the statistics of random process $\hat{n}(y)$. However, its analysis is very difficult and some simplification is desirable. To this end, we will assume that the random process $\hat{n}(y)$ deviates by a small amount from a random variable, i.e.

$$\hat{n}(y) = \hat{n} + \varepsilon\hat{v}(y) \tag{43}$$

where $\hat{n}$ is a random variable, $\hat{v}(y)$ is a random process and ε is a small parameter. Note that since $\hat{n}$ is not taken to be slightly random, equation (43) does not imply that $\hat{n}(y)$ is slightly random.

We may now use ε as a perturbation parameter in the analysis of equation (42), and accordingly we expand the stress in a perturbation series

$$\frac{\hat{\sigma}_2}{\sigma_c} = \hat{s}_0 + \varepsilon\hat{s}_1 + \varepsilon^2\hat{s}_2 + \ldots. \tag{44}$$

Inserting equations (43) and (44) into equation (42), expanding the integrand in a Taylor series in ε, and then grouping terms of equal order, we obtain a sequence of equations which may be solved successively for $\hat{s}_0, \hat{s}_1, \hat{s}_2, \ldots$. The result up to order ε^2 is given as

$$\begin{aligned}\frac{\hat{\sigma}_2}{\sigma_c} = {} & \frac{1}{1+2^{(\hat{n}-2)/2\hat{n}}} - \varepsilon\frac{(\ln 2)2^{(\hat{n}-2)/2\hat{n}}}{\hat{n}^2(1+2^{(\hat{n}-2)/2\hat{n}})^2}\left[\frac{1}{l}\int_0^l \hat{v}(y)\,\mathrm{d}y\right] \\ & + \varepsilon^2\left\langle\frac{(\ln 2)\ln[2(1+2^{(\hat{n}-2)/2\hat{n}})^{2\hat{n}}]}{\hat{n}^3(1+2^{(\hat{n}-2)/2\hat{n}})^2 2^{(\hat{n}+2)/2\hat{n}}}\left[\frac{1}{l}\int_0^l \hat{v}(y)^2\,\mathrm{d}y\right]\right. \\ & \left. - \frac{(\ln 2)\{[1+2^{(\hat{n}-2)/2\hat{n}}][\hat{n}\ln(1+2^{(\hat{n}-2)/2\hat{n}})-1]+\ln 2\}2^{(\hat{n}-2)/2\hat{n}}}{\hat{n}^3(1+2^{(\hat{n}-2)/2\hat{n}})^3}\left[\frac{1}{l}\int_0^l \hat{v}(y)\,\mathrm{d}y\right]^2\right\rangle.\end{aligned} \tag{45}$$

We have indicated previously that if $\hat{N}(y)$ is a homogeneous random process with an exponential-type autocorrelation, then $\hat{n}(y)$ is a homogeneous random process which is ergodic in the mean. Assuming further that $E\{\hat{n}(y)\} = E\{\hat{n}\}$ we obtain for $l \to \infty$ the result $E\{\hat{v}(y)\} = (1/l)\int_0^l \hat{v}(y)\,\mathrm{d}y = 0$, and equation (45) simplifies to

$$\frac{\hat{\sigma}_2}{\sigma_c} = \frac{1}{1+2^{(\hat{n}-2)/2\hat{n}}} + \varepsilon^2\frac{(\ln 2)\ln[2(1+2^{(\hat{n}-2)/2\hat{n}})^{2\hat{n}}]}{\hat{n}^3(1+2^{(\hat{n}-2)/2\hat{n}})^2 2^{(\hat{n}+2)/2\hat{n}}}\sigma_v^2 \tag{46}$$

where σ_v^2 is the variance of $\hat{v}(y)$ when $\hat{n}(y)$ is taken to be ergodic in the autocorrelation. Given that $\hat{n}(y)$ deviates by order ε from the random variable $\hat{n}$, equation (46) indicates that the stress $\hat{\sigma}_2$ deviates by order ε^2 from the solution for $\hat{n}(y) = \hat{n}$. Hence, it appears that an analysis of the special case $\hat{n}(y) = \hat{n}$ will yield reasonably accurate statistical information about $\hat{\sigma}_2$. In the remainder of this section we will confine our attention to this special case, for which we set $\varepsilon = 0$ in equation (46) and obtain

$$\hat{\sigma}_2 = \frac{\sigma_c}{1+2^{(\hat{n}-2)/2\hat{n}}} \tag{47}$$

in agreement with equation (15a).

For convenience, we rewrite equation (47) in nondimensional form as

$$\frac{\hat{\sigma}_2}{\sigma_{20}} = \frac{g(n_0)}{1+2^{(\hat{n}-2)/2\hat{n}}} \tag{48}$$

where the nominal stress σ_{20} and the constant $g(n_0)$ are given by

$$\sigma_{20} = \frac{\sigma_c}{g(n_0)}, \qquad g(n_0) = 1+2^{(n_0-2)/2n_0}. \tag{49}$$

Given that $\hat{n}$ is lognormal [equation (29)], the density function for $\hat{\sigma}_2/\sigma_{20}$ follows from equation (48) as

$$f\left(\frac{\sigma_2}{\sigma_{20}}\right) = \frac{g(n_0)}{(\sigma_2/\sigma_{20})[g(n_0)-(\sigma_2/\sigma_{20})]\sigma_N n_0'\sqrt{(2\pi)}\ln 2} \cdot \frac{[1+(n_0-1)X]^2\, e^{-a^2(\ln X)^2}}{X}\left[U\left(\frac{\sigma_2}{\sigma_{20}}-\frac{g(n_0)}{1+\sqrt{2}}\right)-U\left(\frac{\sigma_2}{\sigma_{20}}-\frac{g(n_0)}{1+1/\sqrt{2}}\right)\right] \tag{50}$$

where

$$X = \frac{\left[\dfrac{\ln 2}{\ln\{\sqrt{(2)}(\sigma_2/\sigma_{20})/[g(n_0)-(\sigma_2/\sigma_{20})]\}}\right]-1}{n_0-1}, \qquad a = \frac{n_0-1}{\sqrt{(2)}\sigma_N n_0'}. \tag{51}$$

Equation (50) has been plotted in Fig. 4 (solid lines) for typical values of n_0 and $\sigma_N n_0'$. The lower cutoff on stress $[\sigma_2/\sigma_{20} = g(n_0)/(1+\sqrt{2})]$ corresponds with $n \to \infty$, while the

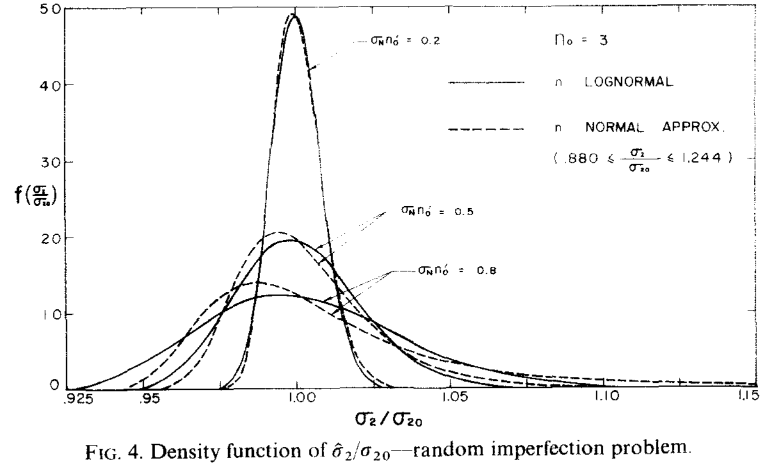

FIG. 4. Density function of $\hat{\sigma}_2/\sigma_{20}$—random imperfection problem.

upper cutoff $[\sigma_2/\sigma_{20}) = g(n_0)/(1+1/\sqrt{2})]$ corresponds with $n = 1$. At both of these points $f(\sigma_2/\sigma_{20})$ equals zero. Note that although the density function for $\hat{n}$ is asymmetric, the density function for $\hat{\sigma}_2/\sigma_{20}$ is almost symmetric with respect to $\sigma_2/\sigma_{20} = 1$. In fact, it is exactly symmetric when $n_0 = 2$.

Due to the complexity of equation (50), we were not able to obtain closed form expressions for the mean and standard deviation of $\hat{\sigma}_2/\sigma_{20}$. However, these statistical properties are easily obtained by numerical integration, and typical curves are shown in Fig. 5. Note

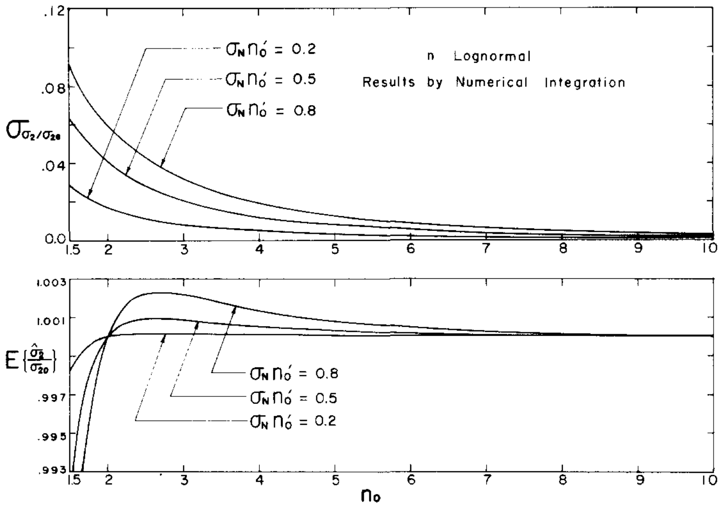

FIG. 5. Mean and standard deviation of $\hat{\sigma}_2/\sigma_{20}$—random imperfection problem.

that, for values of the material parameters within the range of practical interest, $E\{\hat{\sigma}_2/\sigma_{20}\}$ is very close to unity and $\sigma_{\sigma_2/\sigma_{20}}$ is very small. For example, consider the rather extreme values $\sigma_N = 10^2\,\text{cm}^{-2}$, $n_0 = 3$ and $n_0' = 0{\cdot}8\times10^{-2}\,\text{cm}^2$; then equations (30) yield $E\{\hat{n}\} = 3{\cdot}167$ and $\sigma_n = 0{\cdot}902$, and Fig. 5 yields $E\{\hat{\sigma}_2/\sigma_{20}\} = 1{\cdot}002$ and $\sigma_{\sigma_2/\sigma_{20}} = 0{\cdot}032$. Thus, although the creep power $\hat{n}$ may fluctuate greatly as a result of random fluctuations in the imperfection density $\hat{N}(y)$, the stress ratio $\hat{\sigma}_2/\sigma_{20}$ is almost insensitive to such fluctuations. We conclude that in the random imperfection problem the stress may be assumed to be deterministic with little loss of accuracy.

For small enough values of $\sigma_N n_0'/(n_0-1)$ (e.g. $\leq 0{\cdot}2$), the above analysis may be simplified. In such cases we may approximate the lognormal distribution for $\hat{n}$ [equation (29)] by a normal distribution [equation (33)]. We then obtain in place of equation (50) the approximate expression

$$f\left(\frac{\sigma_2}{\sigma_{20}}\right) \approx \frac{g(n_0)\ln 2}{\sqrt{(2\pi)}\sigma_n\left(\dfrac{\sigma_2}{\sigma_{20}}\right)\left[g(n_0)-\left(\dfrac{\sigma_2}{\sigma_{20}}\right)\right]\left[\ln\left(\dfrac{\sqrt{(2)}(\sigma_2/\sigma_{20})}{g(n_0)-(\sigma_2/\sigma_{20})}\right)\right]^2}$$
$$\exp\left\{\frac{-1}{2\sigma_n^2}\left[\frac{\ln 2}{\ln\left(\dfrac{\sqrt{(2)}(\sigma_2/\sigma_{20})}{g(n_0)-(\sigma_2/\sigma_{20})}\right)}-n_0\right]^2\right\},\quad n \text{ normal.} \tag{52}$$

This equation has also been plotted in Fig. 4 (dashed lines). Note that the approximate curve is very close to the exact curve for $\sigma_N n_0'/(n_0-1) = 0{\cdot}1$, whereas a sizable asymmetric error occurs for $\sigma_N n_0'/(n_0-1) = 0{\cdot}4$. The interval outside of $g(n_0)/(1+\sqrt{2}) \leq \sigma_2/\sigma_{20} \leq g(n_0)/(1+1/\sqrt{2})$ has no physical meaning, and portions of the approximate curves in this region should be ignored.

Finally, after expanding equation (48) in a Taylor series about n_0, we may obtain approximate closed form expressions for the statistical moments (see [6]). Again assuming $\sigma_N n_0'/(n_0-1)$ small, we may truncate this series after the second order term and utilize the statistical moments of the normal approximation to $\hat{n}$ to obtain

$$E\left\{\frac{\hat{\sigma}_2}{\sigma_{20}}\right\} \approx 1+\frac{2^{-1/n_0}(\ln 2)[2n_0 g(n_0)-(\ln 2)\bar{g}(n_0)]\sigma_n^2}{\sqrt{(2)}n_0^4 g(n_0)^2} \tag{53a}$$

$$\sigma_{\sigma_2/\sigma_{20}} \approx \frac{\sqrt{(2)}(2^{-1/n_0})(\ln 2)\sigma_n}{n_0^2 g(n_0)} \tag{53b}$$

where

$$\bar{g}(n_0) = 1-2^{(n_0-2)/2n_0}. \tag{54}$$

Table 1 gives several values of the statistical moments as calculated by numerical integration (Fig. 5) and by the approximate Taylor series formulas [equations (53)] with $\sigma_N n_0' = 0{\cdot}5$. Note that $\hat{\sigma}_2/\sigma_{20}$ becomes less random as n_0 increases.

TABLE 1. MEAN AND STANDARD DEVIATION OF $\hat{\sigma}_2/\sigma_{20}$, $\sigma_N n_0' = 0{\cdot}5$

	$E\{\hat{\sigma}_2/\sigma_{20}\}$		$\sigma_{\sigma_2/\sigma_{20}}$	
n_0	Numerical integration	Taylor series	Numerical integration	Taylor series
3	1·00085	1·00342	0·02019	0·02037
5	1·00029	1·00077	0·00766	0·00765
7	1·00012	1·00029	0·00398	0·00397
10	1·00004	1·00010	0·00198	0·00197

2. *Velocity solution.* Considering the velocity now, we obtain from equations (14b) [with $\hat{\dot{\varepsilon}}_c(y) = \dot{\varepsilon}_{c0}$)] in conjunction with equations (46) and (49) the relations

$$\left(\frac{\hat{\dot{v}}}{\dot{v}_0}\right) = \frac{1}{l}\int_0^l \hat{\phi}(y)\,\mathrm{d}y \tag{55a}$$

$$\dot{v}_0 = \dot{\varepsilon}_{c0}\left(\frac{\sigma_{20}}{\sigma_c}\right)^{n_0} l \tag{55b}$$

$$\hat{\phi}(y) = \{H[\hat{n}(y)]\}^{\hat{n}(y)} g(n_0)^{n_0} \tag{55c}$$

where $\dot{v}_0$ is the nominal velocity and $H[\hat{n}(y)]$ represents the right hand side of equation (46). Here, we have again assumed that $\hat{n}(y)$ deviates by a small amount from a random variable [equation (43)].

Equation (55a) is similar in form to equation (36a) and proceeding in the same manner we obtain for the mean and variance of $\hat{\dot{v}}/\dot{v}_0$ the expressions

$$E\{\hat{\dot{v}}/\dot{v}_0\} = E\{\hat{\phi}(y)\} \tag{56a}$$

$$\sigma^2_{\dot{v}/\dot{v}_0} = \frac{1}{l}\int_0^{2l}\left(1-\frac{\eta}{2l}\right)C_\phi(\eta)\,d\eta \tag{56b}$$

where $C_\phi(\eta)$ is the autocovariance of $\hat{\phi}(y)$. After specifying the autocovariance of $\hat{N}(y)$ [e.g. equation (32)], one may evaluate the autocovariance of $\hat{n}(y)$ from equation (31), and then use equations (43), (46) and (55c) to find $C_\phi(\eta)$. Equations (56) then yield the mean and the variance of the velocity.

The procedure described above will in general involve extensive numerical calculations. Rather than pursue this course further, we will restrict our attention to the special case $\hat{N}(y)$ and $\hat{n}(y)$ equal to random variables. As we learned in the random temperature problem, this is a conservative assumption since it yields an upper bound on the variance [see equations (41)]. Equations (46) and (51) yield in this case

$$\frac{\hat{\dot{v}}}{\dot{v}_0} = \hat{\phi} = \frac{g(n_0)^{n_0}}{[1+2^{(\hat{n}-2)/2\hat{n}}]^{\hat{n}}} \tag{57}$$

which agrees with equation (14b).

Given that $\hat{n}$ is lognormal [equation (29)], we may find the probability density function of $\hat{\dot{v}}/\dot{v}_0$ from equation (57). However, it is obtained in implicit form since equation (57) may not be inverted in closed form. The result is

$$f\left(\frac{\dot{v}}{\dot{v}_0}\right) = \left.\frac{(n_0-1)\exp\left\{-\dfrac{(n_0-1)^2}{2(\sigma_N n_0')^2}\left[\ln\left(\dfrac{n-1}{n_0-1}\right)\right]^2\right\}}{\sqrt{(2\pi)}\sigma_N n_0'(n-1)\left(\dfrac{\dot{v}}{\dot{v}_0}\right)\left[\ln(1+2^{(n-2)/2n})+\dfrac{\ln 2}{n(1+2^{-(n-2)/2n})}\right]}\right|_{n=h(\dot{v}/\dot{v}_0)}$$

$$\cdot\left[U\left(\frac{\dot{v}}{\dot{v}_0}\right)-U\left(\frac{\dot{v}}{\dot{v}_0}-\frac{g(n_0)^{n_0}}{1+1/\sqrt{2}}\right)\right] \tag{58}$$

where $n = h(\dot{v}/\dot{v}_0)$ is the root of the expression

$$\frac{\dot{v}/\dot{v}_0}{g(n_0)^{n_0}}[1+2^{(n-2)/2n}]^n-1 = 0. \tag{59}$$

The lower cutoff $[\dot{v}/\dot{v}_0 = 0]$ corresponds with $n \to \infty$, while the upper cutoff $[\dot{v}/\dot{v}_0 = g(n_0)^{n_0}/(1+1/\sqrt{2})]$ corresponds with $n = 1$.

Equation (58) has been plotted in Fig. 6 (solid lines). The mean and the standard deviation are readily obtained by numerical integration, and some typical curves are given in Fig. 7. The velocity is clearly very sensitive to variations in imperfection density. For example, considering again the values $\sigma_N n_0' = 0{\cdot}8$ and $n_0 = 3$, we obtain from Fig. 7 $E\{\hat{\dot{v}}/\dot{v}_0\} = 1{\cdot}094$ and $\sigma_{\dot{v}/\dot{v}_0} = 0{\cdot}656$. Furthermore, we note from Fig. 7 that the curves level off rapidly and show little dependence on n_0 in the range where experimental values are most likely to fall. Also, in this range the mean velocity exceeds the nominal velocity.

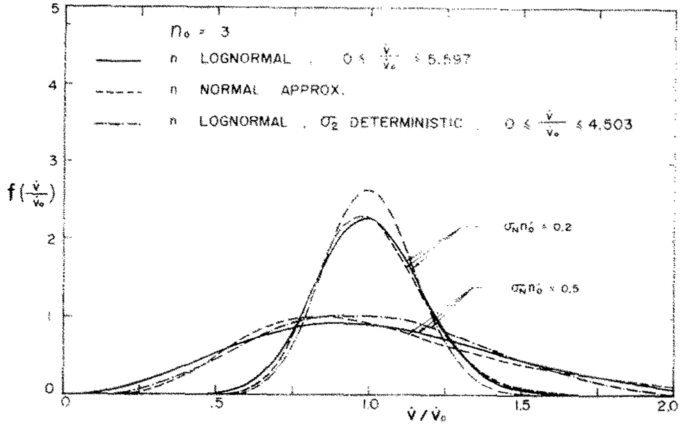

FIG. 6. Density function of $\hat{v}/\dot{v}_0$—random imperfection problem.

As previously demonstrated, we may approximate the lognormal distribution for $\hat{n}$ by a normal distribution when $\sigma_N n_0'/(n_0-1) \leq 0{\cdot}2$. Equation (58) is then replaced by the approximate implicit relation

$$f\left(\frac{\dot{v}}{\dot{v}_0}\right) \approx \left.\frac{\exp\left[-\dfrac{(n-n_0)^2}{2\sigma_n^2}\right]}{\sqrt{(2\pi)}\sigma_n\left(\dfrac{\dot{v}}{\dot{v}_0}\right)\left[\ln(1+2^{(n-2)/2n})+\dfrac{\ln 2}{n(1+2^{-(n-2)/2n})}\right]}\right|_{n=h(\dot{v}/\dot{v}_0)}, \quad n \text{ normal} \tag{60}$$

where $n = h(\dot{v}/\dot{v}_0)$ is defined by equation (59). This simplified equation has been plotted in Fig. 6 (dashed lines). Again we ignore the interval outside of $0 \leq \dot{v}/\dot{v}_0 \leq g(n_0)^{n_0}/(1+1/\sqrt{2})$.

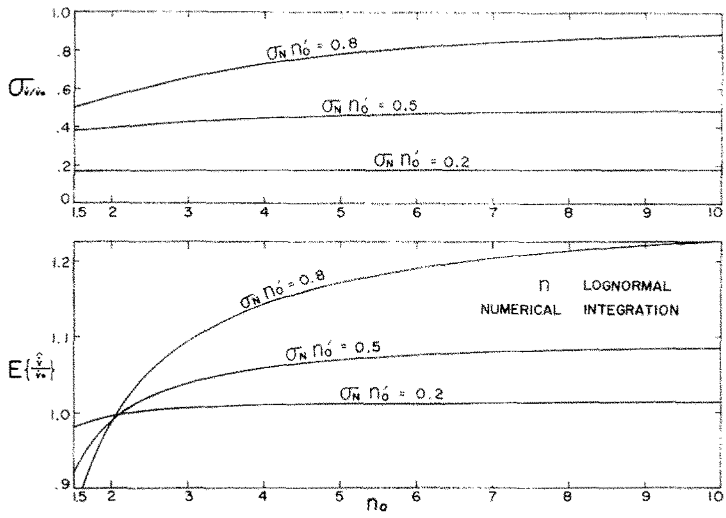

FIG. 7. Mean and standard deviation of $\hat{v}/\dot{v}_0$—random imperfection problem.

Although equation (60) is simpler than equation (58), it is still in an inconvenient implicit form. An alternate simplification may be obtained by utilizing the fact that $\hat{\sigma}_2$ is only slightly random. Thus in equation (48) we assume that σ_2 is deterministic and equal to σ_{20}, whereupon the quantity $1+2^{(\hat{n}-2)/2\hat{n}}$ is also deterministic and equal to $g(n_0)$. Equation (57) then simplifies to

$$\frac{\hat{\dot{v}}}{\dot{v}_0} = g(n_0)^{n_0-\hat{n}}. \tag{61}$$

This equation is easily inverted, and given that $\hat{n}$ is lognormal we obtain the density function in explicit form as

$$f\left(\frac{\dot{v}}{\dot{v}_0}\right) \approx \frac{1}{(\dot{v}/\dot{v}_0)\ln[g(n_0)]}\,\frac{n_0-1}{\sqrt{(2\pi)}\sigma_N n_0'\{n_0-1-\ln(\dot{v}/\dot{v}_0)/\ln[g(n_0)]\}}$$

$$\cdot\exp\left\{-\frac{(n_0-1)^2}{2(\sigma_N n_0')^2}\left[\ln\left(1-\frac{1}{n_0-1}\frac{\ln(\dot{v}/\dot{v}_0)}{\ln[g(n_0)]}\right)\right]^2\right\}\left[U\left(\frac{\dot{v}}{\dot{v}_0}\right)-U\left(\frac{\dot{v}}{\dot{v}_0}-g(n_0)^{n_0-1}\right)\right],$$

$$\sigma_2 \text{ deterministic.} \tag{62}$$

This equation has also been plotted in Fig. 6 (dash–dot lines). Since $\hat{\sigma}_2/\sigma_{20}$ becomes less random as n_0 increases, equation (62) becomes more accurate with greater values of n_0.

Finally, we obtain approximate Taylor series formulas for the mean and variance. Thus, assuming $\sigma_N n_0'/(n_0-1)$ to be small we expand equation (57) in a truncated Taylor series about n_0 and utilize the statistical moments for the normal approximation. By actual numerical evaluation we find that the result shows an extremely small dependence on n_0, and so a considerable simplification is obtained with negligible loss in accuracy by letting $n_0 \to \infty$. Retaining terms up to the fourth order we obtain the result

$$E\left\{\frac{\hat{\dot{v}}}{\dot{v}_0}\right\} \approx 1+\tfrac{1}{2}\sigma_n^2[\ln(\sqrt{(2)}-1)]^2+\tfrac{1}{8}\sigma_n^4[\ln(\sqrt{(2)}-1)]^4 \tag{63a}$$

$$\sigma^2_{\dot{v}/\dot{v}_0} \approx \sigma_n^2[\ln(\sqrt{(2)}-1)]^2+\tfrac{3}{2}\sigma_n^4[\ln(\sqrt{(2)}-1)]^4. \tag{63b}$$

Table 2 gives a comparison of the statistical moments as obtained by numerical integration (Fig. 7) and by approximate formulas (63) for $\sigma_N n_0' = 0{\cdot}5$. Note that the approximate values are rather crude for $n_0 = 3$ $[\sigma_N n_0'/(n_0-1) = 0{\cdot}25]$, but they improve rapidly as n_0 increases.

TABLE 2. MEAN AND STANDARD DEVIATION OF $\hat{\dot{v}}/\dot{v}_0$, $\sigma_N n_0' = 0{\cdot}5$

	$E\{\hat{\dot{v}}/\dot{v}_0\}$		$\sigma_{\dot{v}/\dot{v}_0}$	
n_0	Numerical integration	Taylor series	Numerical integration	Taylor series
3	1·03893	1·10182	0·42548	0·50077
5	1·06823	1·10182	0·45880	0·50077
7	1·07894	1·10182	0·47355	0·50077
10	1·08637	1·10182	0·48462	0·50077

5. SUMMARY OF RESULTS

We have presented significant statistical results pertaining to steady creep in a 3-bar truss, where the load is deterministic while the material parameters $\dot{\varepsilon}_c$ and n are described in a stochastic sense. The randomness in these parameters results from randomness in the temperature and imperfection density, which have been assumed to be independent homogeneous normal random processes that are equal in the three bars. The problem has been separated into two uncoupled parts—the "random temperature problem" and the "random imperfection problem". In the random temperature problem only $\dot{\varepsilon}_c$ is random, whereas in the random imperfection problem only n is random. It has been shown that both of these parameters are described by lognormal probability density functions. Experimental evidence indicates that large fluctuations in both parameters are to be expected, and this fact has guided the analysis.

The statistics of the stress and velocity have been found for the two problems. In the random temperature problem, the stress is deterministic, whereas the velocity is very sensitive to variations in temperature. We find that the mean velocity always exceeds the nominal velocity, and that the statistics of the velocity ratio (velocity divided by nominal velocity) are independent of n. Similar results are found in the random imperfection problem. Here the stress is found to be only very slightly random, whereas the velocity is again found to be highly random as the imperfection density fluctuates. For values of the parameters of greatest practical interest, we find that the mean velocity again exceeds the nominal velocity and that the statistics of the velocity ratio exhibits only a minor dependence on the nominal value of n. In both problems, simple and useful formulas are presented for the statistical properties of the stress and velocity.

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Абстракт—Даётся анализ касающийся эффекта безпорядочных параметров материала и его влияния на постоянную ползучесть в трёхстерженевой ферме.

Параметр безпорядочности вообще большой.

Это приводится благодаря беспорядчности по отношению температуры и плотности неточностей. Даются аналитические и числённые результаты статических свойств параметров материала и свойств напряжении и скорости. Указано, что беспорядочность в параметрах будет с одной стороны вызывать очень малую беспорядочность ь маираюсемиах, но с другой стороны очень значительную по отномении к скорости.